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Anomalous Scaling in a Model of Passive Scalar Advection: Exact Results

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Abstract

Kraichnan's model of passive scalar advection in which the driving velocity field has fast temporal decorrelation is studied as a case model for understanding the appearance of anomalous scaling in turbulent systems. We demonstrate how the techniques of renormalized perturbation theory lead (after exact resummations) to equations for the statistical quantities that reveal also non perturbative effects. It is shown that ultraviolet divergences in the diagrammatic expansion translate into anomalous scaling with the inner length acting as the renormalization scale. In this paper we compute analytically the infinite set of anomalous exponents that stem from the ultraviolet divergences. Notwithstanding, non-perturbative effects furnish a possibility of anomalous scaling based on the outer renormalization scale. The mechanism for this intricate behavior is examined and explained in detail. We show that in the language of L'vov, Procaccia and Fairhall [Phys. Rev. E **50**, 4684 (1994)] the problem is "critical" i.e. the anomalous exponent of the scalar primary field $\Delta = \Delta_c$. This is precisely the condition

that allows for anomalous scaling in the structure functions as well, and we prove that this anomaly must be based on the outer renormalization scale. Finally, we derive the scaling laws that were proposed by Kraichnan for this problem, and show that his scaling exponents are consistent with our theory.

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1. INTRODUCTION

The model of passive scalar advection with rapidly decorrelating velocity field which was introduced some time ago by R.H. Kraichnan [1] was suggested recently [2] as a case model for understanding multiscaling in the statistical description of nonlinear field theories. The model is for a scalar field $\Theta(\mathbf{r},t)$ where $\mathbf{r}$ is a point in R^d. This field satisfies the equation of motion

$$[\partial_t - \kappa\nabla^2 + \mathbf{u}(\mathbf{r},t)\cdot\boldsymbol{\nabla}]\Theta(\mathbf{r},t) = f(\mathbf{r},t). \tag{1.1}$$

In this equation $f(\mathbf{r},t)$ is the forcing and $\mathbf{u}(\mathbf{r},t)$ is the velocity field which is taken to be a stochastic field, rapidly varying in time. The forcing is taken to be delta correlated in time and space-homogeneous,

$$\langle f(\mathbf{r},t)f(\mathbf{r}',t')\rangle = \Phi_0(\mathbf{r}-\mathbf{r}')\delta(t-t'). \tag{1.2}$$

We will study this model in the limit of large Peclet (Pe) number, which is defined as the dimensionless ratio $U_L L/\kappa$, where U_L is the scale of the velocity fluctuations on the outer scale L of the system. The most important property of the driving velocity field from the point of view of the scaling properties of the passive scalar is the tensor "eddy diffusivity" [1]

$$\begin{aligned} h_{ij}(\mathbf{R}) \equiv \int_0^\infty d\tau \langle [u_i(\mathbf{r}+\mathbf{R},t+\tau) - u_i(\mathbf{r},t+\tau)] \\ \times [u_j(\mathbf{r}+\mathbf{R},t) - u_j(\mathbf{r},t)]\rangle\rangle \end{aligned} \tag{1.3}$$

where the symbol $\langle ... \rangle$ in Eq.(1.3) stands for an ensemble average with respect to the statistics of $\mathbf{u}$ which is given *a priori*. The scaling properties of the scalar depend sensitively on the scaling exponent ζ_h that characterizes the R dependence of $h_{ij}(\mathbf{R})$:

$$h_{ij}(\mathbf{R}) = h(R)\left[(\zeta_h+1)\delta_{ij} - \zeta_h \frac{R_i R_j}{R^2}\right], \tag{1.4}$$

$$h(R) = H\left(\frac{R}{\mathcal{L}}\right)^{\zeta_h}, \tag{1.5}$$

where $\mathcal{L}$ is some characteristic scale of the driving velocity field. The structure of the tensor h_{ij} is dictated by the incompressibility of the velocity field, and H is a free dimensional parameter. By "scaling properties" we mean how various correlation and response functions of $\Theta(\mathbf{r},t)$ and its gradients depend on separation distances. For example the structure functions of $\Theta(\mathbf{r},t)$ are

$$S_{2n}(\mathbf{R}) \equiv \langle [\Theta(\mathbf{r}+\mathbf{R},t) - \Theta(\mathbf{r},t)]^{2n} \rangle \ . \tag{1.6}$$

In writing this equation we assumed that the statistics of the velocity field leads to a stationary and space homogeneous ensemble of the scalar Θ. If the statistics is also isotropic, then S_{2n} becomes a function of R only, independent of the direction of $\mathbf{R}$. The scaling exponents of the structure functions $S_{2n}(R)$ characterize their R dependence in the limit of large Pe,

$$S_{2n}(R) \sim R^{\zeta_{2n}} , \tag{1.7}$$

when R is in the "inertial" interval of scales that will be discussed later in this paper.

In Ref. [2] Kraichnan showed that when the driving velocity field $\mathbf{u}(\mathbf{r},t)$ is delta correlated in time one can derive the exact "balance" equations for the structure functions $S_{2n}(R)$:

$$-\hat{B}(R) S_{2n}(R) = J_{2n}(R) \ . \tag{1.8}$$

In this equation $\hat{B}(R)$ is the linear operator that will be used below repeatedly:

$$\hat{B}(R) = -R^{1-d} \frac{\partial}{\partial R} R^{d-1} h(R) \frac{\partial}{\partial R} , \tag{1.9}$$

with d being the space dimension. On the RHS of the balance equations we have

$$J_{2n}(R) = 4n\kappa \left\langle [\Theta(\mathbf{r}+\mathbf{R}) - \Theta(\mathbf{r})]^{2n-2} |\nabla\Theta(\mathbf{r})|^2 \right\rangle \ . \tag{1.10}$$

Kraichnan conjectured that the scaling dependence of $J_{2n}(R)$ on R when R is in the inertial range is given by the law

$$J_{2n}(R) = J_2(R)S_{2n}(R)/S_2(R) \sim R^{\zeta_{2n}-\zeta_2} \ . \tag{1.11}$$

This scaling law led Kraichnan to far reaching conclusions regarding the scaling exponents ζ_{2n}. Once inserted in the balance equations this conjecture resulted in the prediction that the scaling exponents satisfy the equation

$$\zeta_{2n}(\zeta_{2n} - \zeta_2 + d) = nd\zeta_2 \ . \tag{1.12}$$

If so, this model may be the first nonlinear non-equilibrium case where "multiscaling" can be explicitly demonstrated. Further, Kraichnan et al. proposed in [3] that if this model *is* multiscaling, then (1.12) is the unique solution for ζ_{2n}. We will see that among other things our considerations lead to precisely the scaling law $J_{2n}(R) \sim R^{\zeta_{2n}-\zeta_2}$ as conjectured by Kraichnan. We will see how the possibility of multiscaling is indeed realised due to the analytic structure of the theory.

In a previous work on this model [4] (referred to hereafter as paper I) renormalized perturbation theory was employed to study the scaling behavior. It is appropriate to present first a short summary of the results of this paper.

The main point of paper I was that various statistical quantities exhibit anomalous exponents that stem from ultraviolet divergences in their diagrammatic expansion. For example there exists a quantity called the nonlinear Green's function (that is defined in sec.2 and is a 4-point quantity) and which is denoted as $\mathcal{G}_2(0|\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4)$. It was shown that for $r_1 \simeq r_2 \simeq r$, $r_3 \simeq r_4 \simeq R$, and $R \gg r$ (cf. I, Eqs.(4.11), (4.12) and (4.18))

$$\frac{\partial}{\partial r_1}\frac{\partial}{\partial r_2}\mathcal{G}_2(0|\mathbf{r}_1, r_2, r_3, r_4) \sim r^{-\Delta} \ , \tag{1.13}$$

where Δ is an anomalous exponent that characterizes the leading divergence when $r \to 0$. Moreover, the value of this exponent is important in determining much of the scaling behavior in this model. It appears prominently in J_{2n} and also in

the correlation function of passive dissipation fluctuations. The dissipation field is defined here as

$$\epsilon(x) \equiv \kappa |\nabla \Theta(x)|^2, \tag{1.14}$$

and the correlation function $K(R)$ is

$$K(R) = \langle \epsilon(\mathbf{r}+\mathbf{R},t)\epsilon(\mathbf{r},t)\rangle - \langle \epsilon(x)\rangle^2 \ . \tag{1.15}$$

It was shown in paper I that the ultraviolet divergences resulted in a dependence on the inner cutoff of the theory, denoted as η [and defined below in Eq. (2.25)] which is written as:

$$K(R)/\kappa^2 \sim \eta^{-2\Delta}. \tag{1.16}$$

It was also explained in paper I that the theory indicates that if Δ reaches the critical value $\Delta_c = \zeta_h$ special considerations must be made. For $\Delta < \zeta_h$ it was shown that the perturbation theory for S_4 and higher order structure functions converges order by order both in the ultra-violet and in the infra-red limits. Accordingly, there is no external renormalization length scale in the theory and there is no (perturbative) mechanism to ruin simple scaling, i.e. $\zeta_{2n} = n\zeta_n$. In this case the correlation of dissipation fluctuations can be shown to decay in the inertial range of scales as

$$K(R) \sim \langle \epsilon(x)\rangle^2 \left(\frac{\eta}{R}\right)^{2\zeta_h - 2\Delta} \ . \tag{1.17}$$

Indeed, as long as $\Delta < \zeta_h$ the correlation decays in the inertial range as it must for a mixing field. On the contrary, if $\Delta = \zeta_h$ Eq.(1.17) cannot continue to hold since it predicts that the correlations do not decay. This is precisely where the need for anomalous scaling of the structure functions comes in. We will see below that the value of Δ is precisely ζ_h, and we will get instead of Eq.(1.17) the following prediction:

$$K(R) \sim \langle \epsilon(x) \rangle^2 \left(\frac{L}{R}\right)^{2\zeta_2 - \zeta_4} \quad . \tag{1.18}$$

Here L is the outer renormalization scale for this model, that in general is not identical with the outer scale $\mathcal{L}$ of Eq.(1.5). This result is in agreement with Kraichnan's conjectures. The detailed explanation of how this phenomenon takes place is one of the major aims of this paper. It is an important mechanism that indicates how at least in this example ultraviolet divergences coupled with nonperturbative effects may conspire together to give at the end anomalous corrections which are carried by the outer renormalization scale. Whether or not such a mechanism operates in other hydrodynamic systems will be discussed in separate publications. At any rate, the example treated here serves as a powerful demonstration of the fact that renormalized perturbation theory as applied to field theories of the hydrodynamic type allows one, after proper resummations, to capture subtle nonperturbative effects.

It needs to be stressed that the anomalous exponent Δ discussed above is just the leading divergence associated with scalar anomalous fields. We will explain below that there exists a full spectrum of anomalous exponents which are associated with the inner length, and they have to do with anomalous fields of different irreducible representation of the rotation group [7]. We will compute below analytically the whole spectrum of these exponents. For the model at hand this is an easy task, but it serves to demonstrate the rich scaling properties of hydrodynamic systems. This rich scaling structure has not been considered by the fluid mechanics community until now.

The structure of this paper is as follows: section 2 is devoted to the derivation of the differential equations satisfied by the n-point time correlations functions and the n-point simultaneous correlation functions, as well as the equations for the 2-point and 4-point Green's functions. The derivation is based on the diagrammatic expansion of paper I, and the main objective of this section is to

demonstrate that this technique yields the exact equations, and that the resulting equations contain aspects of the problem which are not perturbative. The results of the calculations of this section are identical to alternative derivations which are based on standard stochastic methods. Readers who are not interested in the method of derivation can start to read this paper from section 3 which begins with a catalog of the equations that are analyzed in the rest of the paper. Section 3 begins with the exact solution of the Green's function and the 2-point correlation function. These solutions are important since they introduce the homogeneous solutions of the operator $\hat{\mathcal{B}}(R)$ which then appears importantly in all the solutions of the higher order quantities. Section 2 B deals with the exact solution of the 2-point correlation function. This solution depends on the nature of the forcing. We are able to study in detail how the effects of nonisotropic forcing on the large scales decay in the inertial range as the scale of observation decreases. The exponents that govern the "law of isotropization" are important in determining the scaling behavior of correlation functions of anomalous fields whose presentation under the symmetry groups is different from scalars. Finally, in section 2 C we discuss the 4-point nonlinear Green's function which allows us the evaluation of the anomalous exponent Δ. Section 4 is devoted to the calculation of $K(R)$, $J_4(R)$, and other correlations that expose non-scalar anomalous fields. The section is based on analyzing the equation for $\mathcal{F}_4$, and the strategy is to extract the leading divergence that is characterized by the anomalous exponent Δ. Section 5 presents a brief calculation of J_{2n} and section 6 collects all the result together in order to compute the scaling exponents of $S_{2n}(R)$. We show that there are only two possibilities, one is simple scaling, and the other is anomalous scaling in agreement with Kraichnan's conjectures. Simple scaling is possible only if the dissipation field is not mixing, which seems a nonphysical condition. Section 7 offers a summary of the paper.

2. DIAGRAMMATIC DERIVATION OF THE EQUATIONS FOR CORRELATION AND RESPONSE FUNCTIONS

In this chapter we demonstrate that for this model the exact resummation of the diagrammatic expansion for the various n-point correlation and response functions of the theory results in exact differential equations that contain both perturbative and nonperturbative anomalies. The analysis is based on paper I, in which the first step is the Belinicher-L'vov transformation [4–6]. This is done by allowing the center of the coordinate system to move along the Lagrangian trajectory of a particular fluid point. The reference point is at position $\mathbf{r}_0$ at time t_0. The trajectory of this point with respect to $\mathbf{r}_0$ is

$$\boldsymbol{\rho}(\mathbf{r}_0, t_0|t) = \int_{t_0}^{t} dt' \, \mathbf{u}(\mathbf{r}_0 + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t'), t'). \tag{2.1}$$

Let us denote the transformed variables as

$$T(\mathbf{r}_0, t_0|\mathbf{r}, t) = \Theta(\mathbf{r} + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t), t), \tag{2.2}$$

$$\mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t) = \mathbf{u}(\mathbf{r} + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t), t), \tag{2.3}$$

$$\phi(\mathbf{r}_0, t_0|\mathbf{r}, t) = f(\mathbf{r} + \boldsymbol{\rho}(\mathbf{r}_0, t_0|t), t). \tag{2.4}$$

In terms of these new variables the equation of motion (1.1) reads

$$\left[\partial_t - \kappa\nabla^2 - \mathbf{w}(\mathbf{r}_0, t_0|\mathbf{r}, t)\cdot\boldsymbol{\nabla}\right] T(\mathbf{r}_0, t_0|\mathbf{r}, t) = \phi(\mathbf{r}_0, t_0|\mathbf{r}, t), \tag{2.5}$$

where

$$\mathbf{w}(\mathbf{r}_0, t_0|\mathbf{r}, t) = \mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}, t) - \mathbf{v}(\mathbf{r}_0, t_0|\mathbf{r}_0, t). \tag{2.6}$$

We turn now to the discussion of the statistical quantities that are defined in terms of these variables.

A. The 2-point quantities

The 2-point functions discussed here are the 2-point correlation function

$$\mathcal{F}(\mathbf{r}_0|x_1, x_2) \equiv \langle T(x_0|x_1)T(x_0|x_2)\rangle \tag{2.7}$$

and the Green's function that is defined as

$$\mathcal{G}(\mathbf{r}_0|x_1, x_2) \equiv \left\langle \frac{\delta T(x_0|x_1)}{\delta\xi(x_0|x_2)} \right\rangle \Bigg|_{\xi\to 0}, \tag{2.8}$$

where the $d+1$ dimensional vector $x \equiv (\mathbf{r}, t)$, and ξ is the Belinicher-L'vov transformation of a forcing that is added to the RHS of Eq.(2.5). The dependence on t_0 disappeared from the LHS of Eqs.(2.8) because one can prove [7] that all the average quantities are time translationally invariant.

In this subsection we present the derivation of the equations for $\mathcal{F}(\mathbf{r}_0|x_1, x_2)$ and $\mathcal{G}(\mathbf{r}_0|x_1, x_2)$. In paper I we derived the Dyson-Wyld equations for these two point functions. For $t > 0$ they read

$$\begin{aligned}
(\partial_t - \kappa\nabla^2)\mathcal{G}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, t) &= \delta(\mathbf{r}_1 - \mathbf{r}_2)\delta(t) \\
&+ \int d\mathbf{r}' \Sigma(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}')G(\mathbf{r}_0|\mathbf{r}', \mathbf{r}_2, t) \qquad (2.9)\\
\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, t) &= \int d\mathbf{r}' d\mathbf{r}'' \int_0^\infty dt' \mathcal{G}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}', t + t') \\
&\times [\Phi_0(\mathbf{r}', \mathbf{r}'') + \Phi(\mathbf{r}', \mathbf{r}'')]\mathcal{G}(\mathbf{r}_0|\mathbf{r}_2, \mathbf{r}'', t'). \qquad (2.10)
\end{aligned}$$

For negative times the Green's function is zero, and the correlation function is symmetric to inverting time $t \to -t$ and the coordinates $\mathbf{r} \to -\mathbf{r}$ together. In the Dyson equation (2.9) the mass operator $\Sigma(\mathbf{r}_0|\mathbf{r}, \mathbf{r}')$ can be written explicitly

$$\Sigma(\mathbf{r}_0|\mathbf{r}, \mathbf{r}') = \frac{\partial}{\partial r_i}\mathcal{G}(\mathbf{r}_0|\mathbf{r}, \mathbf{r}', t = 0)H_{ij}(\mathbf{r}_0|\mathbf{r}, \mathbf{r}')\frac{\partial}{\partial r'_j} \tag{2.11}$$

whereas in the Wyld equation (2.10) the mass operator Φ takes the form

$$\Phi(\mathbf{r}_0|\mathbf{r}, \mathbf{r}') = H_{ij}(\mathbf{r}_0|\mathbf{r}, \mathbf{r}')\frac{\partial}{\partial r_i}\frac{\partial}{\partial r'_j}\mathcal{F}(\mathbf{r}_0|\mathbf{r}, \mathbf{r}', t = 0). \tag{2.12}$$

In (2.11) and (2.12)

$$H_{ij}(\mathbf{r}_0|\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} dt \langle w_i(\mathbf{r}_0, t_0|\mathbf{r}, t)w_j(\mathbf{r}_0, t_0|\mathbf{r}', 0)\rangle. \tag{2.13}$$

In the case considered in this paper in which the velocity field is delta correlated in time the expression (2.13) can be related to the eddy diffusivity (1.3), expressed in terms of Eulerian correlations as:

$$H_{ij}(\mathbf{r}_0|\mathbf{r},\mathbf{r}') = h_{ij}(\mathbf{r}-\mathbf{r}_0) + h_{ij}(\mathbf{r}'-\mathbf{r}_0) - h_{ij}(\mathbf{r}-\mathbf{r}'). \tag{2.14}$$

That the mass operators (2.11) and (2.12) have an explicit form corresponding to the 1-loop diagram rather than an infinite series representation stems from the fact that the velocity field decorrelates on an infinitely short time scale. This simple form of the mass operators will be lost if we relax this fast decay of the velocity time correlation functions.

Equations (2.9) and (2.10) can be turned into differential equations for the 2-point functions $\mathcal{F}$ and $\mathcal{G}$. Substituting (2.11) in (2.9) we find that

$$\begin{aligned}(\partial_t - \kappa\nabla^2)\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t) &= \delta(\mathbf{r}-\mathbf{r}')\delta(t) \\ +\int d\mathbf{r}''\frac{\partial}{\partial r_i}\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}'',0)&H_{ij}(\mathbf{r}_0|\mathbf{r},\mathbf{r}'')\frac{\partial}{\partial r''_j}G(\mathbf{r}_0|\mathbf{r}'',\mathbf{r}',t).\end{aligned} \tag{2.15}$$

Remember that $\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t)$ is zero for negative times. Integrating (2.15) over time from $t=-\delta$ to $t=\delta$ and taking the limit $\delta\to 0$ one finds that

$$\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t=0^+) = \delta(\mathbf{r}-\mathbf{r}')\ . \tag{2.16}$$

We will choose symmetrical regularization at $t=0$ and by convention write

$$\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t=0) = \frac{1}{2}\delta(\mathbf{r}-\mathbf{r}'). \tag{2.17}$$

Using this evaluation the integration may be performed leading to

$$\left[\partial_t + \hat{\mathcal{D}}_1(\mathbf{r}-\mathbf{r}_0)\right]\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t) = \delta(\mathbf{r}-\mathbf{r}')\delta(t). \tag{2.18}$$

Here we introduced the generalized diffusion operator

$$\hat{\mathcal{D}}_1(\mathbf{r}) = -\kappa\nabla^2 + \hat{\mathcal{B}}(\mathbf{r})\ , \tag{2.19}$$

where the operator $\hat{\mathcal{B}}$ is a key operator that appears repeatedly below. We will distinguish between an operator $\hat{\mathcal{B}}(\mathbf{r}_\alpha, \mathbf{r}_\beta)$ which acts on functions of two variables $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$ and an operator $\hat{\mathcal{B}}(\mathbf{R})$ acting on functions of one variable $\mathbf{R}$:

$$\hat{\mathcal{B}}(\mathbf{r}_\alpha, \mathbf{r}_\beta) \equiv \hat{\mathcal{B}}_{\alpha,\beta} = h_{ij}(\mathbf{r}_\alpha - \mathbf{r}_\beta) \frac{\partial^2}{\partial r_{\alpha,i} \partial r_{\beta,j}} , \tag{2.20}$$

$$\hat{\mathcal{B}}(\mathbf{R}) = -h_{ij}(R) \frac{\partial^2}{\partial R_i \partial R_j} . \tag{2.21}$$

Clearly $\hat{\mathcal{B}}(\mathbf{r}_\alpha, \mathbf{r}_\beta)$ turns into $\hat{\mathcal{B}}(\mathbf{R})$ on the class of functions depending on the difference $\mathbf{R} = \mathbf{r}_\alpha - \mathbf{r}_\beta$ only. In spherical coordinates the $\hat{\mathcal{B}}$-operator can be represented as the sum of two contributions:

$$\hat{\mathcal{B}}(\mathbf{R}) = \hat{B}(R) + \frac{(\zeta_h + d - 1)}{(d-1)} \frac{h(R)}{R^2} \hat{L}^2 . \tag{2.22}$$

Here $\hat{B}$ is the operator (1.9) and $\hat{L}$ is the angular momentum operator $-i\mathbf{r} \times \boldsymbol{\nabla}$ which depends only on the direction of $\mathbf{r}$.

For future purposes we also need the equation of the time-integrated Green's function $\mathcal{G}(\mathbf{r} - \mathbf{r}_0, \mathbf{r}' - \mathbf{r}_0)$ which is defined as

$$\mathcal{G}(\mathbf{r} - \mathbf{r}_0, \mathbf{r}' - \mathbf{r}_0) = \int dt \mathcal{G}(\mathbf{r}_0 | \mathbf{r}, \mathbf{r}', t). \tag{2.23}$$

This function satisfies the equation

$$\hat{\mathcal{D}}_1(\mathbf{R}) \mathcal{G}(\mathbf{R}, \mathbf{R}') = \delta(\mathbf{R} - \mathbf{R}') , \tag{2.24}$$

which follows from Eq. (2.18).

The equation of motion (2.24) allows us to introduce the inner scale of this model, denoted by η. By definition η is the scale for which the advective term $\hat{\mathcal{B}}$ is of the order of the dissipative term $\kappa \nabla^2$. Equating the two terms we get

$$\eta \simeq \left(\frac{\kappa}{H} \right)^{1/\zeta_h} . \tag{2.25}$$

We proceed now to determine an equation of motion for the two-point correlator $\mathcal{F}(\mathbf{r}_0 | \mathbf{r}_1, \mathbf{r}_2, t)$. It is clear from Eq. (2.18) that one may define an inverse operator for the Green's function $\mathcal{G}(\mathbf{r}_0 | \mathbf{r}, \mathbf{r}', t)$ according to

$$\mathcal{G}_1^{-1}(\mathbf{r}-\mathbf{r}_0,t) \equiv \partial_t - \hat{\mathcal{D}}_1(\mathbf{r}-\mathbf{r}_0) \ . \tag{2.26}$$

Note from Eq. (2.18) that the equation of motion for $\mathcal{G}(\mathbf{r}_0|\mathbf{r},\mathbf{r}',t)$ only depends on the first coordinate $\mathbf{r}$, so that $\mathcal{G}_1^{-1} \equiv \mathcal{G}_1^{-1}(\mathbf{r}-\mathbf{r}_0,t)$. Operating with $\mathcal{G}^{-1}$ we may rewrite the Wyld equation (2.10) as

$$\begin{aligned}&\left[\partial_t + \hat{\mathcal{D}}_1(\mathbf{r}_1-\mathbf{r}_0)\right]\left[-\partial_t + \hat{\mathcal{D}}_1(\mathbf{r}_2-\mathbf{r}_0)\right]\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t)\\ &= \delta(t)\left[\Phi(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2) + \Phi_0(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2)\right],\end{aligned} \tag{2.27}$$

where we have used the fact that $\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t)$ is only a function of the time difference $t = t_1 - t_2$. We have also the "boundary" condition

$$\mathcal{F}(\mathbf{r}_0|\mathbf{r}_1,\mathbf{r}_2,t) \to 0 \quad \text{for} \quad t \to \pm\infty \ . \tag{2.28}$$

Next we want to derive the differential equation satisfied by the simultaneous 2 point correlator $\mathcal{F}(\mathbf{R})$:

$$\mathcal{F}(\mathbf{R}) = \mathcal{F}(\mathbf{r}_1,\mathbf{r}_2,t=0) \tag{2.29}$$

where $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$. The derivation is described in detail in Appendix A with the final result

$$\hat{\mathcal{D}}_2(\mathbf{R})\mathcal{F}(\mathbf{R}) = \Phi_0(\mathbf{R}), \tag{2.30}$$

where generally the operator $\hat{\mathcal{D}}_2(\mathbf{r}_\alpha,\mathbf{r}_\beta)$ operates on two coordinates:

$$\hat{\mathcal{D}}_2(\mathbf{r}_\alpha,\mathbf{r}_\beta) = -\kappa[\nabla_\alpha^2 + \nabla_\beta^2] + \hat{\mathcal{B}}(\mathbf{r}_\alpha,\mathbf{r}_\beta) \ , \tag{2.31}$$

but in the case where the operand is a function only of the difference $\mathbf{R} = \mathbf{r}_\alpha - r_\beta$ reduces to

$$\hat{\mathcal{D}}_2(\mathbf{R}) \equiv -2\kappa\nabla^2 + \hat{\mathcal{B}}(\mathbf{R}) \ . \tag{2.32}$$

B. The derivation of the differential equations for higher order correlations and response functions

In this subsection we present the equations for the 4-point and higher order correlation functions and for the nonlinear Green's function (2.8). The solutions are deferred to the next section.

1. The 4-point Green's function

In paper I we presented the diagrammatic series for the nonlinear Green's function $\mathcal{G}_2(\mathbf{r}_0|x_1, x_2, x_3, x_4)$. This quantity is defines as

$$\mathcal{G}_2(\mathbf{r}_0|x_1, x_2, x_3, x_4) \equiv \left\langle \frac{\delta T(x_0|x_1)}{\delta \xi(x_0|x_3)} \frac{\delta T(x_0|x_2)}{\delta \xi(x_0|x_4)} \right\rangle \Bigg|_{\xi \to 0} . \tag{2.33}$$

The diagrammatic expansion of this quantity is an infinite series of ladder diagrams that can be resummed exactly. In Fig. 1 we recall the notation for the diagrammatic elements, and in Fig. 2 is reproduced the diagrammatic resummed equation for this function. In analytic form this equation reads

$$\begin{aligned} \mathcal{G}_2(0|x_1, x_2, x_3, x_4) &= \mathcal{G}_2^{\mathrm{G}}(0|x_1, x_2, x_3, x_4) \\ + \int d\mathbf{r}' d\mathbf{r}'' &\int_{t_m}^{\infty} dt' \, \mathcal{G}_2^{\mathrm{G}}(0|x_1, x_2, \mathbf{r}', t', \mathbf{r}'', t') \\ \times H_{ij}(\mathbf{r}', \mathbf{r}'') &\frac{\partial}{\partial r_i'} \frac{\partial}{\partial r_j''} G_2(0|\mathbf{r}', t', \mathbf{r}'', t', x_3, x_4) \end{aligned} \tag{2.34}$$

where $t_m = \min\{t_1, t_2\}$ and

$$\mathcal{G}_2^{\mathrm{G}}(0|x_1, x_2, x_3, x_4) \equiv \mathcal{G}(0|x_1, x_3)\mathcal{G}(0|x_2, x_4). \tag{2.35}$$

We want now to derive a differential equation for this quantity. We can proceed as in the case of the correlation function by applying the product of the inverse operators $\tilde{\mathcal{G}}_1^{-1}$ of Eq. (2.26) to obtain a differential equation for $\mathcal{G}_2(0|x_1, x_2, x_3, x_4)$ as a function of three time differences. We may however make use of the structure of Eq. (2.34) to directly derive an equation in only one time difference. The

availability of such a reduction follows from the delta-correlated velocity field in much the same way as the availability of a differential operator for $\mathcal{F}(0|\mathbf{r}_1, \mathbf{r}_2, t = 0)$ as we saw before. We can choose at will to consider Eq. (2.34) for the times $t_1 = t_2 = t$ and $t_3 = t_4 = 0$ to derive a differential equation in one time difference for the quantity

$$\mathcal{G}_2(0|\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, t) \equiv \mathcal{G}_2(0|\mathbf{r}_1, t, \mathbf{r}_2, t, \mathbf{r}_3, 0, \mathbf{r}_4, 0) \tag{2.36}$$

The derivation resembles closely the derivation of the equation for the two-point correlator described in Appendix A. Applying the operator (2.27) to Eq. (2.34) with the above choice of times we get

$$\begin{aligned} & [\partial_t + \hat{\mathcal{D}}_2(\mathbf{r}_1, \mathbf{r}_2) + \hat{\mathcal{H}}(\mathbf{r}_1, \mathbf{r}_2)]\mathcal{G}_2(0|\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, t) \\ = & \ \delta(\mathbf{r}_1 - \mathbf{r}_3)\delta(\mathbf{r}_2 - \mathbf{r}_4)\delta(t) , \end{aligned} \tag{2.37}$$

where $\hat{\mathcal{D}}_2(\mathbf{r}_1, \mathbf{r}_2)$ is the operator defined in Eq. (2.32) and $\hat{\mathcal{H}}$ is an operator defined in Appendix A that arises from the derivation of the equation for the two-point correlator.

For the time integrated 4-point Green's function

$$\mathcal{G}_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \int dt \mathcal{G}_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4, t) \tag{2.38}$$

one has the equation

$$\left[\hat{\mathcal{D}}_2(\mathbf{r}_1, \mathbf{r}_2) + \hat{\mathcal{H}}(\mathbf{r}_1, \mathbf{r}_2)\right] \mathcal{G}_2(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) = \delta(\mathbf{r}_1 - \mathbf{r}_3)\delta(\mathbf{r}_2 - \mathbf{r}_4) \ . \tag{2.39}$$

The non-linear Greens' function is the kernel of the response to some forcing at points $\mathbf{r}_3$ and $\mathbf{r}_4$, and we are interested in this response. In order to study this, we introduce a new function

$$\Psi(\mathbf{r}_1, \mathbf{r}_2) = \int d\mathbf{r}_3 d\mathbf{r}_4 \mathcal{G}(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) A(\mathbf{r}_3, \mathbf{r}_4) \ , \tag{2.40}$$

where $A(\mathbf{r}_3, \mathbf{r}_4)$ is an arbitrary function. From Eq.(2.39) one may determine that

$$\left[\hat{\mathcal{D}}_2(\mathbf{r}_1,\mathbf{r}_2)+\hat{\mathcal{H}}\right]\Psi(\mathbf{r}_1,\mathbf{r}_2)=A(\mathbf{r}_1,\mathbf{r}_2). \tag{2.41}$$

If one chooses now to restrict $A(\mathbf{r}_1,\mathbf{r}_2)$ to the space of functions depending only upon the difference $\mathbf{R}=\mathbf{r}_1-\mathbf{r}_2$, the equation simplifies (as in the derivation in Appendix A) to the form

$$\hat{\mathcal{D}}_2(\mathbf{R})\Psi(\mathbf{R})=A(\mathbf{R}). \tag{2.42}$$

We will return to an analysis of this equation in Sec.3C.

2. The simultaneous higher order correlations

The equations of motion for the time dependent $2n$-th order correlation functions are derived in Appendix B. Here we will derive the equations for the simultaneous correlations. The simultaneous $2n$-point correlator is

$$\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{2n})=\langle T(0|\mathbf{r}_1,t)T(0|\mathbf{r}_2,t),...,T(0|\mathbf{r}_{2n},t)\rangle\,. \tag{2.43}$$

This same time quantity is identical in the Eulerian frame of reference and in the transformed reference frame that we use. We can thus forget the $\mathbf{r}_0$ designation, and remember that in homogeneous systems the quantity is a function only of differences of its space arguments. Its time derivative is on the one hand zero and on the other hand

$$\begin{aligned}&\frac{\partial}{\partial t}\mathcal{F}_{2n}(\mathbf{r}_1,\mathbf{r}_2,...,\mathbf{r}_{2n}) \qquad (2.44)\\ &=\sum_{\alpha=1}^{2n}\left\langle T(0|\mathbf{r}_1,t)\ \ldots\ \frac{\partial}{\partial t}T(0|\mathbf{r}_\alpha,t)\ \ldots\ T(0|\mathbf{r}_{2n},t)\right\rangle\,.\end{aligned}$$

Using the equation of motion (2.5) we find that the RHS of (2.44) has three types of terms, one with $\mathbf{w}$ (advection), one with κ (dissipation) and the last proportional to the force (forcing). These terms are denoted as $A_{\rm adv}$, $A_{\rm dis}$ and $A_{\rm for}$ with

$$A_{\rm adv} + A_{\rm dis} + A_{\rm for} = 0 \tag{2.45}$$

where

$$A_{\rm adv} = \sum_{\alpha=1}^{2n} \nabla_\alpha \cdot \mathbf{F}_{w,2nT}(\mathbf{r}_\alpha, \mathbf{r}_1 ... \mathbf{r}_{2n}) \tag{2.46}$$

and

$$\mathbf{F}_{w,2nT}(\mathbf{r}_\alpha, \mathbf{r}_1 \ \ldots \ \mathbf{r}_{2n}) \tag{2.47}$$

$$= \langle \mathbf{w}_\alpha T(0|\mathbf{r}_1, t) \ldots T(0|\mathbf{r}_{\alpha-1}, t) T(0|\mathbf{r}_\alpha, t) \ldots T(0|\mathbf{r}_{2n}, t) \rangle$$

$$A_{\rm dis} = \kappa \sum_{\alpha=1}^{2n} \nabla_\alpha^2 \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, \ \ldots \ \mathbf{r}_{2n}) \ . \tag{2.48}$$

Finally

$$A_{\rm for} = \sum_{\alpha=1}^{2n} \langle T(0|\mathbf{r}_1, t) ... T(0|\mathbf{r}_{\alpha-1}, t) \phi(0|\mathbf{r}_\alpha, t) ... T(0|\mathbf{r}_{2n}, t) \rangle \ . \tag{2.49}$$

The diagrammatic representation of $\mathbf{F}_{w,2nT}(\mathbf{r}_\alpha, \mathbf{r}_1 ... \mathbf{r}_{2n})$ was discussed in paper I with a final result [cf. paper I Eq. (5.7)] which in $\mathbf{r}, t$ representation is

$$\mathbf{F}_{w,2nT}(\mathbf{r}_\alpha, \mathbf{r}_1 ... \mathbf{r}_{2n}) \tag{2.50}$$

$$= \frac{1}{2} \sum_{\beta=1}^{2n} \mathbf{H}(0|\mathbf{r}_\alpha, \mathbf{r}_\beta) \cdot \nabla_\beta \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n})$$

Introducing this result into (2.47) and using (2.14) we find

$$A_{\rm adv} = \frac{1}{2} \sum_{\alpha,\beta=1}^{2n} [h_{ij}(\mathbf{r}_\alpha) + h_{ij}(\mathbf{r}_\beta) - h_{ij}(\mathbf{r}_\alpha - \mathbf{r}_\beta)]$$

$$\times \frac{\partial^2}{\partial r_{\alpha i} \partial r_{\beta j}} \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n}). \tag{2.51}$$

This form can be rewritten equivalently as

$$A_{\rm adv} = \Bigg\{ - \sum_{\alpha > \beta=1}^{2n} \hat{\mathcal{B}}_{\alpha\beta} \tag{2.52}$$

$$+ \left[\sum_{\alpha=1}^{2n} h_{ij}(\mathbf{r}_\alpha) \frac{\partial}{\partial r_{\alpha i}} \right] \left[\sum_{\beta=1}^{2n} \frac{\partial}{\partial r_{\beta j}} \right] \Bigg\} \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n}).$$

Finally we use the fact that $\mathcal{F}_{2n}$ is a function of differences of its spatial arguments to see that the second operator in the first line of the RHS gives zero and the line can be omitted. Thus

$$A_{\rm adv} = -\sum_{\alpha>\beta=1}^{2n} \hat{\mathcal{B}}_{\alpha\beta}\mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n}). \tag{2.53}$$

Consider next the forcing term. Using the fact that for a Gaussian force

$$\begin{aligned}
&\langle T(0|\mathbf{r}_1,t)...T(0|\mathbf{r}_{\alpha-1},t)\phi(0|\mathbf{r}_\alpha,t)...T(0|\mathbf{r}_{2n},t)\rangle \\
&= \int \left\langle \frac{\delta T(0|\mathbf{r}_1,t)...T(0|\mathbf{r}_{\alpha-1},t)...T(0|\mathbf{r}_{2n},t)}{\delta\phi(0|\mathbf{r}'_\alpha,t')} \right\rangle \\
&\times \Phi_0(\mathbf{r}_\alpha - \mathbf{r}'_\alpha)d\mathbf{r}'_\alpha
\end{aligned} \tag{2.54}$$

where we used (1.2) and the fact that we deal with simultaneous correlations. The functional derivative in the integrand is a zero time response, which as usual is computed in the non-interacting limit:

$$\begin{aligned}
&\left\langle \frac{\delta T(0|\mathbf{r}_1,t)...T(0|\mathbf{r}_{\alpha-1},t)...T(0|\mathbf{r}_{2n},t)}{\delta\phi(0|\mathbf{r}'_\alpha,t')} \right\rangle \\
&= \sum_\beta G^0(\mathbf{r}_\alpha - \mathbf{r}_\beta, t=0)\mathcal{F}_{2n-2}(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_{2n})
\end{aligned} \tag{2.55}$$

where in $\mathcal{F}_{2n-2}(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_{2n})$ the two arguments $\mathbf{r}_\alpha$ and $\mathbf{r}_\beta$ are missing. Substituting (2.55) in (2.54) and the result in (2.49), we find

$$A_{\rm for} = \sum_{\alpha>\beta} \Phi_0(\mathbf{r}_\alpha - \mathbf{r}_\beta)\mathcal{F}_{2n-2}(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_{2n}). \tag{2.56}$$

Substituting all these results in (2.45) yields

$$\begin{aligned}
&\left[-\kappa\sum_\alpha \nabla^2_\alpha + \sum_{\alpha>\beta}^{2n} \hat{\mathcal{B}}_{\alpha\beta}\right] \mathcal{F}_{2n}(\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_{2n}) \\
&= \sum_{\alpha>\beta} \Phi_0(\mathbf{r}_\alpha - \mathbf{r}_\beta)\mathcal{F}_{2n-2}(\mathbf{r}_1, \mathbf{r}_2, ...\mathbf{r}_{2n}) \ .
\end{aligned} \tag{2.57}$$

This equation, which is our final equation for the simultaneous correlation function, is again identical with Kraichnan's.

3. Equation for the irreducible correlation $\mathcal{F}_4^{\rm c}$

The irreducible (cumulant) part of the correlation functions are defined as

$$\mathcal{F}^{\rm c}_{2n} = \mathcal{F}_{2n} - \mathcal{F}^G_{2n}. \tag{2.58}$$

The equation for the simultaneous $\mathcal{F}_4^c$ follows from specializing Eq. (2.57) to the case $2n = 4$, using Eq. (2.30) for $\mathcal{F}_2$,

$$\begin{aligned}
&\left[-\sum_{\alpha}^{4} \kappa \nabla_\alpha^2 + \sum_{\alpha>\beta} \hat{\mathcal{B}}_{\alpha\beta}\right] \sum_{\alpha>\beta=1}^{4} \mathcal{F}_4^c(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3, \mathbf{r}_4) \\
&= -\frac{1}{2} \sum_{\substack{(\alpha_i) \\ \text{perm}(1234)}} \hat{\mathcal{B}}(\mathbf{r}_{\alpha_1} - \mathbf{r}_{\alpha_3}) \mathcal{F}_2(\mathbf{r}_1, \mathbf{r}_2) \mathcal{F}_2(\mathbf{r}_3, \mathbf{r}_4) \\
&= -\Big[(\hat{\mathcal{B}}_{14} + \hat{\mathcal{B}}_{23} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{24}) \mathcal{F}_{12} \mathcal{F}_{34} \\
&+ (\hat{\mathcal{B}}_{14} + \hat{\mathcal{B}}_{23} + \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34}) \mathcal{F}_{13} \mathcal{F}_{24} \\
&+ (\hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{24} + \hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34}) \mathcal{F}_{14} \mathcal{F}_{23}\Big] \; .
\end{aligned} \tag{2.59}$$

3. ANALYSIS OF THE GREEN'S FUNCTIONS AND THE 2-POINT CORRELATION

In this section we begin to discuss the exact solution for the various quantities in this model when all the separation distances are in the inertial interval. For the sake of clarity we catalogue here all the equations that we are going to solve below, neglecting the diffusive terms which appear in the full equations. The equations are the following, for

1) the time integrated Green's function cf.(2.23):

$$\hat{\mathcal{B}}(\mathbf{R}) \mathcal{G}(\mathbf{R}, \mathbf{R}') = \delta(\mathbf{R} - \mathbf{R}') \; , \tag{3.1}$$

which follows from (2.24),

2) the 2 point simultaneous correlation functions,

$$\hat{\mathcal{B}}(\mathbf{R}) \mathcal{F}(\mathbf{R}) = \Phi(\mathbf{R}) \, , \tag{3.2}$$

which follows from (2.30),

3) the function Ψ introduced in Sec.2B,

$$\hat{\mathcal{B}}(\mathbf{R}) \Psi(\mathbf{R}) = A(\mathbf{R}), \tag{3.3}$$

which is the $\kappa \to 0$ limit of Eq.(2.42),

4) the cumulant of the 4-point simultaneous correlation function

$$
\begin{aligned}
&\Big[\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{13}+\hat{\mathcal{B}}_{14}+\hat{\mathcal{B}}_{23}+\hat{\mathcal{B}}_{24}+\hat{\mathcal{B}}_{34}\Big]\mathcal{F}^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)\\
&= -\Big[\hat{\mathcal{B}}_{14}+\hat{\mathcal{B}}_{23}+\hat{\mathcal{B}}_{13}+\hat{\mathcal{B}}_{24}\Big]\mathcal{F}(\mathbf{r}_1-\mathbf{r}_2)\mathcal{F}(\mathbf{r}_3-\mathbf{r}_4)\\
&- \Big[\hat{\mathcal{B}}_{14}+\hat{\mathcal{B}}_{23}+\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}\Big]\mathcal{F}(\mathbf{r}_1-\mathbf{r}_3)\mathcal{F}(\mathbf{r}_2-\mathbf{r}_4)\\
&- \Big[\hat{\mathcal{B}}_{13}+\hat{\mathcal{B}}_{24}+\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}\Big]\mathcal{F}(\mathbf{r}_1-\mathbf{r}_4)\mathcal{F}(\mathbf{r}_2-\mathbf{r}_3)\ ,
\end{aligned}
\tag{3.4}
$$

which is the $\kappa \to 0$ limit of eq.(2.59).

It is noteworthy that the same operator $\hat{\mathcal{B}}$, (2.20), (2.22) appears in all these equations. Indeed, the most interesting scaling properties will be seen to arise from the eigenfunctions of $\hat{\mathcal{B}}$ with zero eigenvalue which are the solutions of the homogeneous equation (3.5).

A. Solutions of the basic homogeneous equation

For two distinct coordinates $\mathbf{R} \neq \mathbf{R}'$ the Green's function $\mathcal{G}(\mathbf{R},\mathbf{R}')$ satisfies the homogeneous part of the equation (3.1)

$$\hat{\mathcal{B}}(\mathbf{R})\mathcal{G}(\mathbf{R},\mathbf{R}') = 0\ , \tag{3.5}$$

which will be referred to hereafter as the " basic homogeneous equation". In light of the representation of Eq.(2.22) of the $\hat{\mathcal{B}}$ operator via the angular momentum operator we represent the solution of (3.5) in 3-dimensional space as an expansion over spherical harmonics

$$\mathcal{G}(\mathbf{R},\mathbf{R}') = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l} b_{ll',mm'}\, g_{lm}(\mathbf{R}) g_{l'm'}(\mathbf{R}') \tag{3.6}$$

where $b_{ll',mm'}$ are coefficients and

$$g_{lm}(\mathbf{R}) = R^{\beta_l} Y_{lm}(\theta,\phi)\ . \tag{3.7}$$

The spherical harmonics $Y_{lm}(\theta,\phi)$ are the eigenfunctions of the angular momentum operator:

$$\hat{L}^2 Y_{lm}(\theta,\phi) = l(l+1)Y_{lm}(\theta,\phi) \tag{3.8}$$

and θ and ϕ are the polar and azimuthal angles of $\mathbf{R}$. Substituting the expansion (3.6)into eq.(3.5) we find the relationship

$$\beta_l(\beta_l + 3 - \zeta_2) - l(l+1)(1+\zeta_h/2) = 0 \ . \tag{3.9}$$

Of the two solutions of this quadratic equation we must select the nonnegative branch since the negative branch is unphysical:

$$\beta_l = \frac{1}{2}\left[\zeta_2 - 3 + \sqrt{(\zeta_2-3)^2 + 2l(l+1)(2+\zeta_h)}\right] \ . \tag{3.10}$$

In 2 dimensions one has a similar representation with the difference that instead of the spherical harmonics we have the eigenfunctions $\exp(il\phi)$ and instead of the eigenvalues $l(l+1)$ we have the eigenvalues l^2. The final result for the exponents β_l in 2- dimensions is

$$\beta_l = \frac{1}{2}\left[-\zeta_h + \sqrt{\zeta_h^2 + 4l^2(1+\zeta_h)}\right] \ . \tag{3.11}$$

Multipole expansions of the type (3.6) and the scaling exponents β_l play an important role in all our development below. We will see that the basic homogeneous equation reappears in various guises, and in each of them we will need to fix the coefficients $b_{ll',mm'}$ such as to respect the symmetry of the particular object involved, the boundary conditions, etc. For example in this case we know that the Green's function $\mathcal{G}(\mathbf{R},\mathbf{R}')$ is symmetric in $\mathbf{R}$ and $\mathbf{R}'$. Accordingly $b_{ll',mm'} = b_{l'l,m'm}$. If the solution depends on R, R' and the angle between $\mathbf{R}$ and $\mathbf{R}'$, then $b_{ll',mm'} \neq 0$ only for $l = l'$, $m + m' = 0$, etc.

B. Exact solution of the 2-point correlation function

1. Isotropic forcing

The solution of the 2-point correlation function depends on the nature of the external forcing $f(\mathbf{r},t)$. It is customary to take $f(\mathbf{r},t)$ to be Gaussian and statistically homogeneous in space and time. The properties of the correlation function $\Phi_0(\mathbf{r}-\mathbf{r}')$ were not determined up to now. Since we are interested in the universal scaling properties of the scalar field we want to choose the forcing such that it has only large scale components. Otherwise the scaling exponent of the 2-point correlation may be coloured by the functional dependence of the forcing on r. On the other hand the forcing may be isotropic or non isotropic. In this subsection we will deal with the isotropic case, and the nonistropic case will be treated in the next subsection.

The properties of $f(\mathbf{r},t)$ are best stated in $\mathbf{k}$-space: it is concentrated in the small k region, i.e. $k \le 1/L$, and it decays quickly to zero for $k \ll 1/L$. In $\mathbf{r}$ space this means that $\langle f(\mathbf{r},t) f(\mathbf{r}+\mathbf{R},t)\rangle$ is constant for $\mathbf{R} \ll L$:

$$\langle f(\mathbf{r},t) f(\mathbf{r}+\mathbf{R},t)\rangle = \Phi_0. \quad (R \ll L) \tag{3.12}$$

Using this form of the forcing correlation function we realize that the inhomogeneous term on the RHS of Eq.(2.29) is translationally and rotationally invariant. Accordingly we can seek solutions that have the same symmetry, solving the equation

$$\hat{B}(R)\mathcal{F}(R) = \Phi_0 \ . \tag{3.13}$$

The solution of the inhomogeneous equation is

$$\mathcal{F}_{\rm inh}(R) = C_{\rm inh} R^{\zeta_2} \ , \tag{3.14}$$

with $C_{\rm inh} = -\Phi_0/2h\zeta_2$ and

$$\zeta_2 = 2 - \zeta_h \ . \tag{3.15}$$

Of course, the inhomogeneous solution has to be supplemented with the solutions of the homogeneous equation in order to match the boundary conditions (which in this case are $\mathcal{F}(R) = \mathcal{F}(0) \neq 0$ at $r = 0$). There are two homogeneous solutions: one is a constant which must be taken as $\mathcal{F}(0)$, and the other is $\mathcal{F}(R) \propto R^{-\zeta_h}$ which is ruled out by the boundary condition $\mathcal{F}(0) < \infty$. One should note that the constant homogeneous solution belongs to the family of exponents β_l with $l = 0$ as one would expect. The solution that is ruled out is indeed the $l = 0$ member of the forbidden branch of solutions of the quadratic equation (3.9).

2. The law of isotropization

When the forcing is nonisotropic on the large scales we need to apply the full operator $\hat{\mathcal{B}}(\mathbf{R})$. In the regime $R \ll L$ the forcing is again a constant which is independent of the angles. However, for $R \simeq L$ we expect nonisotropic forcing which leads to nonisotropic tails for every value of R. We can find the law of isotropization of $\mathcal{F}(\mathbf{R})$ for R small by solving the equation that involves now also the nonisotropic part of the operator $\hat{\mathcal{B}}(\mathbf{R})$

$$\hat{\mathcal{B}}(\mathbf{R})\mathcal{F}(\mathbf{R}) = \Phi_0 \ , \quad R \ll L. \tag{3.16}$$

The inhomogeneous solutions are the same as before, but in order to match with the anisotropic behaviour at $R \simeq L$ we need to invoke the whole set of homogeneous solutions $g_{lm}(\mathbf{R})$, Eq. (3.7). The solution in 3 dimensions is written as

$$\mathcal{F}(\mathbf{R}) = \mathcal{F}(0) + R^{\zeta_2}\left[C_{in} + \sum_{l=2}^{\infty}\sum_{m=-l}^{l} a_{lm}\left(\frac{R}{L}\right)^{\beta_l-\zeta_2} Y_{lm}(\theta,\phi)\right]. \tag{3.17}$$

To understand this result we note that all values of β_l are larger than ζ_2. In particular β_2 (for both $d = 3$ and $d = 2$) is

$$\beta_2 = \frac{1}{2}\left[\zeta_2 - d + \sqrt{(\zeta_2 + d)^2 + 24\zeta_h}\right] . \tag{3.18}$$

For positive ζ_h, β_2 is larger than ζ_2. For larger l, β_l is even larger, and dependent on dimension. Thus all the anisotropic terms decay when $R \ll L$. Note that the coefficients a_{lm} are nonuniversal and should be found by matching at $R \sim L$. The law of decay is however universal.

3. Anisotropic structure functions

The solution (3.17) suggests the introduction of anisotropic structure functions via the definition

$$S_{2,lm}(R) = \int d\theta d\phi Y_{lm}(\theta, \phi) S_2(\mathbf{R}) \ . \tag{3.19}$$

Here l should be even due to the symmetry with respect to the inversion of $\mathbf{R}$. These anisotropic structure functions display "clean" scaling behavior with the exponent β_l:

$$S_{2,lm}(R) \sim S_2(L) \left(\frac{R}{L}\right)^{\beta_l} \ . \tag{3.20}$$

We will see that the same scaling exponents feature prominently below.

C. The nonlinear Green's function and the anomalous exponent Δ

In this subsection we discuss the nonlinear Green's function and the anomalous exponent Δ which is associated with it, cf. (1.13). To this aim we return to the equation for Ψ, Eq.(3.3). We realize that the solutions of this equation are identical to those discussed already in the context of the 2-point correlator. For a constant function A we can write the solution

$$\Psi(\mathbf{R}, \mathbf{R}') = \Psi(0) + C|\mathbf{R} - \mathbf{R}'|^{\zeta_2} \ . \tag{3.21}$$

Taking the derivative with respect to $\mathbf{R}$ and $\mathbf{R}'$ as required by Eq.(1.13) we find

$$(\boldsymbol{\nabla}_1 \cdot \boldsymbol{\nabla}_2)\Psi(\mathbf{R}, \mathbf{R}') \sim |\mathbf{R} - \mathbf{R}'|^{-\zeta_h} \tag{3.22}$$

Comparing now with the definition of the anomalous exponent Δ in Eq.(1.13) we see that in the limit $|\mathbf{R}-\mathbf{R}'| \to \eta$ this quantity diverges like $1/\eta^{\zeta_h}$ and

$$\Delta = \zeta_h. \tag{3.23}$$

As explained before this value of Δ is the critical value Δ_c, and the implications of this finding are explored below. We turn now to the appearance of the anomalous exponent in the 4-point correlator and related quantities.

4. THE FOUR POINT CORRELATOR AND RELATED QUANTITIES: $K(R)$, AND $J_4(R)$ AND $L(R)$

In this section we present the analysis of the four point correlator $\mathcal{F}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ and of the quantities that are related it, i.e the correlation of dissipation fluctuation $K(R)$, Eq.(1.15) and the RHS of the balance equation $J_4(R)$, Eq.(1.10). These last quantities are not exactly 4-point correlations, but they are obtained as a limit of $\mathcal{F}_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$. The simpler limit is $K(R)$ which is a centred correlation function and is therefore related to $\mathcal{F}_4(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$:

$$\begin{aligned} K(R) = \kappa^2 & \lim_{r_{12},r_{34}\to 0}\lim_{r_{13}\to R}(\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_2)(\boldsymbol{\nabla}_3\cdot\boldsymbol{\nabla}_4) \\ & \times\left[\mathcal{F}_4^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)+\mathcal{F}_2(\mathbf{r}_1,\mathbf{r}_3)\mathcal{F}_2(\mathbf{r}_2,\mathbf{r}_4)\right]. \end{aligned} \tag{4.1}$$

Since we discovered that the the second derivative is singular , cf. Eq.(3.22), we need to carefully examine the above limit.

The quantity $J_4(R)$ has a Gaussian decomposition $J_4^G(R)$ and a cumulant part $J_4^c(R)$. The Gaussian decomposition is trivially computed as

$$J_4^G(R) = \bar\epsilon S_2(R) + \frac{\kappa}{2}[\nabla S_2(R)]^2 . \tag{4.2}$$

The cumulant part is related to $\mathcal{F}_4^c$ via two terms, $J_4^c(R) = J_{4,1}(R) + J_{4,2}(R)$,

$$J_{4,1}(R) = \kappa\left\langle\left\langle|\boldsymbol{\nabla}_1\Theta(\mathbf{r}_1)|^2\Theta^2(\mathbf{r}_2)\right\rangle\right\rangle , \tag{4.3}$$

$$J_{4,2}(R) = -2\kappa\left\langle\left\langle|\boldsymbol{\nabla}_1\Theta(\mathbf{r}_1)|^2\Theta(\mathbf{r}_1)\Theta(\mathbf{r}_2)\right\rangle\right\rangle , \tag{4.4}$$

where double brackets denote the cumulant part. These two contributions may be considered as the following limits of $\mathcal{F}^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$

$$J_{4,1}(R) = -\kappa \lim_{r_{12},r_{34}\to 0} \lim_{r_{13}\to R} (\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_2)\mathcal{F}_4^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \ , \tag{4.5}$$

$$J_{4,2}(R) = 2\kappa \lim_{r_{12},r_{13}\to 0} \lim_{r_{14}\to R} (\boldsymbol{\nabla}_1\cdot\boldsymbol{\nabla}_2)\mathcal{F}_4^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \ . \tag{4.6}$$

Note that in Eq.(4.5) we have two pairs of coalescing points, i.e 1,2 and 3,4, which are separated by a large distance R. In Eq.(4.6) we have three coalescing points, i.e. 1,2,3 and this group is separated from point 4 by R. One should also note that J_4 is obtained from the full $\mathcal{F}_4$ and not from the cumulant part.

In the next subsections we are going to make strong use of the divergence with respect to small distances. Our strategy will be to expose the leading exponent in the divergence with respect to small separations and to compute it exactly. Then we will find the exponent of the dependence on R by power counting, knowing the overall scaling exponent ζ_4 of $\mathcal{F}_4^c$. In other words, our basic assumption is that the correlator $\mathcal{F}_4^c$ is a homogeneous function of its arguments as long as all of them are in the inertial range:

$$\mathcal{F}_4^c(\lambda\mathbf{r}_1,\lambda\mathbf{r}_2,\lambda\mathbf{r}_3,\lambda\mathbf{r}_4) = \lambda^{\zeta_4}\mathcal{F}_4^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \ , \tag{4.7}$$

where ζ_4 is the unknown scaling exponent that characterizes the structure function $S_4(R)$. We will not make any assumption about the numerical value of ζ_4.

A. Two coalescing pairs of points

1. The effective equation for $\mathcal{F}_4^c$

Consider Eq.(2.59) in the limit $r_{12}, r_{34} \ll R$, but all separations in the inertial interval. This allows us to neglect the diffusion terms, and to conclude that $\hat{\mathcal{B}}_{12} \propto r_{12}^{-\zeta_2}$ and $\hat{\mathcal{B}}_{34} \propto r_{34}^{-\zeta_2}$ are much larger than all the other $\hat{\mathcal{B}}_{\alpha\beta}$ operators which are proportional to $R^{-\zeta_2}$. This suggests to rewrite the equation in the form

$$\Big[\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}+\sum\hat{\mathcal{B}}\Big]\mathcal{F}_4^c(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)=\mathrm{RHS}\ , \tag{4.8}$$

where the sum is on all the four $\hat{\mathcal{B}}$ operators other than the ones explicitly displayed. The calculation of the RHS of Eq.(2.59) is elementary since we know everything explicitly. The result of the calculation is

$$\mathrm{RHS}=R^{\zeta_2}\left[A_0+A_1\left(\frac{r_{12}^{\zeta_h}+r_{34}^{\zeta_h}}{R^{\zeta_h}}\right)+A_2\left(\frac{r_{12}r_{34}}{R^2}\right)^{\zeta_2}\right]\ , \tag{4.9}$$

where A_0, A_1 and A_2 are some dimensionless constants. The last term in the bracket comes from the first group of terms in the RHS of Eq.(2.59). Indeed, the product of two $\mathcal{F}_2$s is proportional to $(r_{12}r_{34})^{\zeta_2}$. The naive evaluation of the sum of $\hat{\mathcal{B}}$'s is $R^{\zeta_h}/r_{12}r_{34}$. However there are two cancellations in the combination of $\hat{\mathcal{B}}$'s which result in the replacement of $r_{12}r_{34}$ by R^2. The sum of $\hat{\mathcal{B}}$'s becomes proportional to $R^{-\zeta_2}$. The leading contribution in the last two groups of terms is regular in r_{12} and r_{34} and therefore can be evaluated as R^{ζ_2}. This is the first term in the bracket. The second term in the bracket is contributed by the sum of $\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}$ in the second and third groups of terms. Higher order terms exist but they are proportional to the second power of the small distances.

To discuss Eq.(4.8) further we note that in a space homogeneous situation $\mathcal{F}_4^c$ is a function of six differences in 3 dimensions and 5 in 2 dimensions. In light of our strategy it is convenient to choose the variables as R, $\mathbf{r}_{12}$ and $\mathbf{r}_{34}$. In doing so we are using seven rather than the needed number of variables, but we will see that the dependence on the extra angle variables disappears. We can now group the RHS of Eq.(4.8) together with $\sum\hat{\mathcal{B}}\mathcal{F}_4^c$ into a new function, say $E(\mathbf{r}_{12},\mathbf{r}_{34},R)$

$$E(\mathbf{r}_{12},\mathbf{r}_{34},R)=\mathrm{RHS}-\sum\hat{\mathcal{B}}\mathcal{F}_4\ . \tag{4.10}$$

In the limits $r_{12},r_{34}\ll R$ we can expand $E(\mathbf{r}_{12},\mathbf{r}_{34},R)$ in orders of r_{12} and r_{34}

$$E(\mathbf{r}_{12},\mathbf{r}_{34},R)=E^{(0)}(R)+E^{(1)}(\mathbf{r}_{12},\mathbf{r}_{34},R)+\dots\ , \tag{4.11}$$

where

$$E^{(0)}(R) = \lim_{r_{12}, r_{34} \to 0} E(\mathbf{r}_{12}, \mathbf{r}_{34}, R) \ . \tag{4.12}$$

The equation that we need to analyze takes on the form

$$\left[\hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34}\right] \mathcal{F}_4^c(\mathbf{r}_{12}, \mathbf{r}_{34}, R) = E^{(0)}(R) + E^{(1)}(\mathbf{r}_{12}, \mathbf{r}_{34}, R) + \dots \ . \tag{4.13}$$

and the explicit expression for $E^{(1)}(\mathbf{r}_{12}, \mathbf{r}_{34}, R)$ will be presented below.

2. Solutions

To leading order we have the effective equation:

$$\left[\hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{34}\right] \mathcal{F}_4^c(\mathbf{r}_{12}, \mathbf{r}_{34}, R) = E^{(0)}(R) \ . \tag{4.14}$$

The leading inhomogeneous solution of Eq.(4.14) is found by inspection:

$$\mathcal{F}^{(1)}_{4,\mathrm{inh}}(\mathbf{r}_{12}, \mathbf{r}_{34}, R) = C_1 E^{(0)}(R)(r_{12}^{\zeta_2} + r_{34}^{\zeta_2}) \ , \tag{4.15}$$

where C_1 is a dimensionless constant. Using the overall scaling exponent ζ_4 that was introduced in Eq.(4.7) we can rewrite this solution as

$$\mathcal{F}^{(1)}_{4,\mathrm{inh}}(\mathbf{r}_{12}, \mathbf{r}_{34}, R) \sim S_4(R) \frac{r_{12}^{\zeta_2} + r_{34}^{\zeta_2}}{R^{\zeta_2}} \ . \tag{4.16}$$

This inhomogeneous solution will be particularly important in the context of the calculation of $J_4(R)$. It does not contribute however to the evaluation of $K(R)$, and for the the latter we need to find the next order inhomogeneous solution. The next order term on the RHS of Eq.(4.13) stems from two sources. First is the A_1 term in Eq.(4.9), and the second is found by substituting (4.16) into $\sum \hat{\mathcal{B}} \mathcal{F}_4^c$ on the RHS of (4.12). Both contributions have the same dependence on r_{12} and r_{34}, and they can be written as:

$$E^{(1)}(r_{12}, r_{34}, R) = E_1(R) \frac{r_{12}^{\zeta_2} + r_{34}^{\zeta_2}}{R^{\zeta_2}} \ . \tag{4.17}$$

Substituting this in Eq.(4.13) produces the next order inhomogeneous solution which reads

$$\mathcal{F}^{(2)}_{4,\text{inh}}(\mathbf{r}_{12},\mathbf{r}_{34},R) = C_1 E^{(1)}(R)\left(\frac{r_{12}r_{34}}{R}\right)^{\zeta_2} \tag{4.18}$$

We can again use Eq.(4.7) to rewrite this solution in the form

$$\mathcal{F}^{(2)}_{4,\text{inh}}(\mathbf{r}_{12},\mathbf{r}_{34},R) \sim S_4(R)\left(\frac{r_{12}r_{34}}{R^2}\right)^{\zeta_2} \ . \tag{4.19}$$

This solution will be shown to give the leading order contribution to $K(R)$, and therefore these orders of the inhomogeneous solution will suffice for our analysis.

In addition to the inhomogeneous solutions all the homogeneous solutions (3.6) found below are available to us, since the homogeneous solutions of $\hat{\mathcal{B}}_{12}$ and of $\hat{\mathcal{B}}_{34}$ are also homogeneous solutions of $\hat{\mathcal{B}}_{12}+\hat{\mathcal{B}}_{34}$. We can therefore write the homogeneous solution as

$$\mathcal{F}_{4,\text{hom}}(\mathbf{r}_{12},\mathbf{r}_{34},R) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\sum_{l'=0}^{\infty}\sum_{m'=-l'}^{l} f_{ll',mm'}(R) g_{lm}(\mathbf{r}_{12}) g_{l'm'}(\mathbf{r}_{34}) \ . \tag{4.20}$$

One should note that at this point we can make a choice of the coordinate system such that the z axis is directed along the $\mathbf{R}$ axis. Accordingly the dependence on the sum of azimuthal angles ϕ_{12} and ϕ_{34} should disappear due to the symmetry of the problem. This requirement is met if all coefficients $f_{ll',mm'}$ vanish when $m+m' \neq 0$. In addition our correlation function is symmetric with respect to the exchange of any pair of points. Accordingly all odd values of l and l' are excluded from the sums in (4.20). Finally we can find the R dependence of these coefficients from the overall power counting. Since $g_{l,m}(\mathbf{r}) \propto r^{\beta_l}$ we can write

$$f_{ll',mm'}(R) \sim S_4(R)/R^{\beta_l+\beta'_l} \ . \tag{4.21}$$

We will need below also the next order inhomogeneous solution which is obtained by substituting (4.20) back into the definition of E (4.10). This solution will be denoted as

$$\mathcal{F}^{(3)}_{4,\text{inh}}(\mathbf{r}_{12},\mathbf{r}_{34},R) \sim \mathcal{F}^{(2)}_{4,\text{hom}}(\mathbf{r}_{12},\mathbf{r}_{34},R)\frac{r_{12}^{\zeta_2}+r_{34}^{\zeta_2}}{R^{\zeta_2}} \ . \tag{4.22}$$

In addition to these homogeneous solutions one can consider also the homogeneous solutions of the sum of the two $\hat{\mathcal{B}}$ operators on the LHS of (4.14). It can be shown that these homogeneous solutions do not add any information that is not contained in the solutions described above.

B. Three coalescing points

For the calculation of J_4 we need also to consider the geometry of three coalescing points, see Eq.(4.4). Accordingly we focus on the limit $r_{12}, r_{13}, r_{23} \ll R \simeq r_{14} \simeq r_{24} \simeq r_{34}$. Instead of Eq.(4.13) we now have the equation

$$\left[\hat{\mathcal{B}}_{12} + \hat{\mathcal{B}}_{13} + \hat{\mathcal{B}}_{23}\right]\mathcal{F}_4^c(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R) = \tilde{E}^{(0)}(R) + \tilde{E}^{(1)}(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R), \tag{4.23}$$

where now the function $\tilde{E}(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R)$ is given by equation (4.10) but with the sum on $\hat{\mathcal{B}}$ containing now one less operator. We seek solutions that are symmetric with respect to permuting the pairs 12, 13, 23. The leading inhomogeneous solution can be found as before, cf.(4.16),

$$\mathcal{F}^{(1)}_{4,\mathrm{inh}}(\mathbf{r}_{12}, \mathbf{r}_{13}, \mathbf{r}_{23}, R) \sim S_4(R)\frac{r_{12}^{\zeta_2} + r_{13}^{\zeta_2} + r_{23}^{\zeta_2}}{R^{\zeta_2}} \ . \tag{4.24}$$

The homogeneous solutions will be of the form (4.20), but with additional summations over l'' and m'' for the third factor $g_{l''m''}$ as a function of the third small vector distance. This is all the information needed for the calculations that follow.

C. Calculation of K(R)

According to our strategy we need now to evaluate the limits shown in Eq.(4.1). We can firstly compute the derivatives operating on the products of $\mathcal{F}_2$ and discover that the limits are not singular. In fact that contributions decays rapidly in R like $R^{-2\zeta_h}$. In computing the derivatives of the cumulant of $\mathcal{F}_4$ we

need to consider all the contributions to the solution of $\mathcal{F}_4$ for two pairs of coalescing points that were examined in subsection 4 A 2, and seek the leading one. The solution (4.15) does not survive the derivative in Eq.(4.1). The $l = l' = 0$ term in (4.20) is constant and also does not survive. The next term that survives the derivatives is the $l = l' = 2$ term, and it is less singular than the inhomogeneous solutions (4.18) since $\beta_2 > \zeta_2$. Accordingly the leading contribution is (4.18), and upon performing the derivatives we get

$$K(R) \sim \frac{S_4(R)}{R^{2\zeta_2}} \lim_{r_{12}, r_{34} \to 0} \frac{\kappa^2}{(r_{12} r_{34})^{\zeta_h}} \ . \tag{4.25}$$

where we have used the fact that $\zeta_h = 2 - \zeta_2$. The singular limit has to be understood in light of the full equation for $\mathcal{F}_4^c$, Eq.(2.59), in which the κ-diffusive terms are explicit. The role of these terms is precisely to truncate the divergence that is implied by (4.25). As a consequence the divergence is only applicable in the inertial range with $r_{12}, r_{34} > \eta$, whereas in the dissipative regime the divergence disappears. Thus for evaluating $K(R)$ via inertial range values we must replace the limit $r_{12}, r_{34} \to 0$ by $r_{12} = r_{34} = \eta$:

$$K(R) \sim \frac{S_4(R)}{R^{2\zeta_2}} \frac{\kappa^2}{\eta^{2\zeta_h}} \ , \tag{4.26}$$

Comparing to Eq.(1.16) we see that we recover the result that $\Delta = \Delta_c = \zeta_h$.

We can rewrite (4.26) in a final form by introducing $\bar{\epsilon}$ as

$$\bar{\epsilon} = \kappa \left\langle |\nabla T|^2 \right\rangle = -\kappa \lim_{r_{12} \to \eta} \nabla_1 \nabla_2 \mathcal{F}(r_{12}) \propto \kappa / \eta^{\zeta_h} \ , \tag{4.27}$$

where we used the fact that $\mathcal{F}(r_{12}) \sim r_{12}^{\zeta_2}$. Using the last equation in the preceding one we find the final result:

$$K(R) \simeq \bar{\epsilon}^2 S_4(R) / S_2(R)^2 \ . \tag{4.28}$$

D. Calculation of the correlation functions $L_{ll',m}$

In this subsection we turn to the calculation of correlation functions that expose the scaling properties of anomalous fields with other irreducible representations of the rotation group. We can do this by introducing the following correlation functions:

$$L_{l,l',m}(r_{12},r_{34},R) = \int d\cos\theta_{12} d\cos\theta_{34} d(\phi_{12}-\phi_{34})$$
$$\times \mathcal{F}_4(\mathbf{r}_{12},\mathbf{r}_{34},R) Y_{l,m}(\theta_{12},\phi_{12}) Y_{l',-m}(\theta_{34},\phi_{34}) \ . \tag{4.29}$$

For $l = l' = 0$ the leading contribution to this correlation function arises from (4.16), and

$$L_{0,0,0}(r_{12},r_{34},R) \simeq \mathcal{F}^{(2)}_{4,\mathrm{inh}} \propto R^{\zeta_4-2\zeta_2} \ . \tag{4.30}$$

For $l,l' \geq 2$ the leading contribution arises from (4.20). Using (4.21) we write the final result

$$L_{l,l',m}(r_{12},r_{34},R) \sim S_4(R) \left(\frac{r_{12}}{R}\right)^{\beta_l} \left(\frac{r_{34}}{R}\right)^{\beta'_l} \ . \tag{4.31}$$

Finally, for $l = 0$ and $l' \geq 2$ or vice versa the leading contribution stems from the solution (4.22). The expression for $L_{0,l',0}(r_{12},r_{34},R)$ can be obtained from (4.31) simply by replacing $\beta_0 = 0$ by ζ_2.

Eq.(4.31) gives rise to a set of anomalous local fields. Taking gradients $\boldsymbol{\nabla}_{1\alpha}\boldsymbol{\nabla}_{2\beta}$, one produces one scalar field $(\boldsymbol{\nabla}_1 \cdot \boldsymbol{\nabla}_2)$, which corresponds to $l = 0$ and has anomalous scaling $\zeta_h = 2 - \zeta_2$, and the traceless tensor $(\boldsymbol{\nabla}_{1\alpha}\boldsymbol{\nabla}_{2\beta} - \frac{1}{3}\boldsymbol{\nabla}_1\boldsymbol{\nabla}_2)$ corresponding to $l = 2$. Taking four gradients one can produce anomalous fields originating from $l = 4$, and so on.

E. Calculation of the cumulant $J^c_4(R)$

The quantity $J_4(R)$ has reducible contributions which were computed above, and a cumulant part which is obtained from $\mathcal{F}^c_4$. The calculation of $J^c_4(R)$ follows

very much the lines of the calculation of $K(R)$, except that one needs to find again which of the solutions found in sec4 A 2 contributes mostly to equations (4.5) and (4.6). It turns out that the leading contribution to (4.5) stems from the solution (4.15), whereas the leading contribution to (4.6) arises from the solution (4.24). As before the derivatives with respect to r_1 and r_2 result in a singular limit when $r_{12} \to 0$, and we have to take $r_{12} = \eta$ after computing the derivatives. On the other hand the other limits are regular and they do not require special care. The evaluation of $J_{4,1}(R)$ and $J_{4,2}(R)$ turn out to be the same up to constants. The result is

$$J_4^c(R) \sim \kappa S_4(R)/R^{\zeta_2}\eta^{\zeta_h} \ . \tag{4.32}$$

Using Eq.(4.27) this can be written finally as

$$J_4^c(R) = C_4 \bar{\epsilon} S_4(R)/S_2(R) \ , \tag{4.33}$$

where C_4 is a dimensionless constant that will be determined below. Together with the reducible contributions that were computed above we can write

$$J_4(R) = \bar{\epsilon} S_2(R) + \frac{\kappa}{2}[\nabla S_2(R)]^2 + C_4 \bar{\epsilon} S_4(R)/S_2(R) \ . \tag{4.34}$$

It is obvious that the second term is small compared with the first and it can be neglected.

5. THE CALCULATION OF $J_{2N}(R)$

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The correlations (1.10) which make $J_{2n}(R)$ have Gaussian decompositions and a cumulant part. The leading contribution to the Gaussian decomposition is

$$J_{2n}^G(R) = \bar{\epsilon} S_{2n-2}(R) \tag{5.1}$$

which corresponds to the first term in (4.34). To evaluate the cumulant part one needs to compute correlation functions of the type

$$J_{p,q}(R) = \Big\langle\Big\langle |\nabla_1\Theta(\mathbf{r}_1)|^2\Theta^{p-2}(\mathbf{r}_1)\Theta^q(\mathbf{r}_1)\Big\rangle\Big\rangle \ . \tag{5.2}$$

with $\mathbf{r}_1 = -\mathbf{R}/2$ and $\mathbf{r}_2 = \mathbf{R}/2$. The calculation of $J_{p,q}(R)$ follows from the equation for $\mathcal{F}^c_{2n}$ upon coalescing a group of p points (denoted below as the α group) into the position $-\mathbf{R}/2$ and a group of q points (denoted as the β group) into the position $\mathbf{R}/2$. We start with Eq.(2.57) which in the inertial interval takes on the form

$$\Big[\sum_{\alpha>\alpha'=1}^{p}\hat{\mathcal{B}}_{\alpha\alpha'} + \sum_{\beta>\beta'=p+1}^{2n}\hat{\mathcal{B}}_{\beta\beta'} + \sum_{\alpha=1}^{p}\sum_{\beta=p+1}^{2n}\hat{\mathcal{B}}_{\alpha\beta}\Big]\mathcal{F}^c_{2n}(\mathbf{r}_1,\dots,\mathbf{r}_{2n}) = RHS \ . \tag{5.3}$$

As before the effective equation is obtained by grouping together a quantity $E(\{r_{\alpha\alpha'}\},\{r_{\beta\beta'}\},R) = RHS - \sum\sum\hat{\mathcal{B}}_{\alpha\beta}\mathcal{F}^c_{2n}(\mathbf{r}_1,\dots,\mathbf{r}_{2n})$. To find the form of the solution we note that when we had one pair of coalescing points this led to the solution (3.14). Two pairs of coalescing points led to (4.16), whereas three pairs of coalescing points resulted in (4.24). In the present case we have $[p(p-1)+q(q-1)]/2$ coalescing pairs and the solution which belongs to the same family is written as

$$\mathcal{F}_{2n,\mathrm{inh}} \sim S_{2n}(R)\Big[\sum_{\alpha>\alpha'=1}^{p} r^{\zeta_2}_{\alpha\alpha'} + \sum_{\beta>\beta'=p+1}^{2n} r^{\zeta_2}_{\beta\beta'}\Big]/R^{\zeta_2} \ . \tag{5.4}$$

Next we need to compute the derivative with respect to $\mathbf{r}_1$ and $\mathbf{r}_2$ and take the limit of all pairs of coalescing distances going to zero. The only divergence will be associated with $r_{12}^{-\zeta_h}$ whereas all the other limits are trivial. According to our strategy we have to cut the divergence at $r_{12} = \eta$ and thus we calculate $\mathcal{F}_{2n,\mathrm{inh}} \sim S_{2n}(R)/\eta^{\zeta_h}$. Using again Eq.(1.17) we have

$$J^c_{2n}(R) = C_{2n}\bar{\epsilon}S_{2n}(R)/S_2(R) \ , \tag{5.5}$$

where C_{2n} is an unknown dimensionless coefficient that will be determined soon. This equation is the generalization of (4.33) for any n. Combining now the cumulant with the leading reducible contribution we get the final result

$$J_{2n}(R) = \bar{\epsilon}\Big[S_{2n-2}(R) + C_{2n}S_{2n}(R)/S_2(R)\Big] \ . \tag{5.6}$$

6. SCALING EXPONENTS OF THE STRUCTURE FUNCTIONS

In this section we collect all the results obtained above with the aim of reaching conclusions about the scaling exponents of the structure functions. The first result that we need to pay attention to is (4.26) or (4.28) for $K(R)$. This result shows that $K(R) \sim R^{\zeta_4-2\zeta_2}$. Since $K(R)$ cannot be an increasing function of R we conclude immediately that

$$\zeta_4 - 2 \le 2\zeta_2 \ . \tag{6.1}$$

Next we need to consider (4.34) for $J_4(R)$, rewriting it after neglecting the second term as

$$J_4(R) \simeq \bar{\epsilon}S_2(R)\Big[1 + C_4\frac{S_4(R)}{S_2^2(R)}\Big]. \tag{6.2}$$

The second term in the parenthesis is dimensionless, and may be written as $(\ell/R)^{2\zeta_2-\zeta_4}$.

Now there are two possibilities:

(i) $\zeta_4 = 2\zeta_2$ and the scaling is normal. The first and third terms have the same scaling and they are of the same order.

(ii) $\zeta_4 < 2\zeta_2$, and the scaling is anomalous. If so, the second term in (6.2) must be larger than the first. For that to happen the renormalization scale *must* be the outer scale L.

The implications of the first possibility are somewhat strange. For example if there is normal scaling the correlation function $K(R)$ does not decay in R. This means that the dissipation field is not mixing. On the contrary, if there is anomalous scaling, then the correlation $K(R)$ decays as is expected from a random field. We will explore now the second possibility and show that if there

is anomalous scaling then the scaling exponents can be computed in agreement with Kraichnan's arguments.

If we accept anomalous scaling, then the leading term in (5.6) is the second one. Generally speaking $J_{2n}(R)$ may be written (following Kraichnan et al. [3]) as

$$J_{2n}(R) = 4n\kappa \int d\Delta\Theta P(\Delta\Theta)[\Delta\Theta]^{2n-2} \left\langle |\nabla\Theta|^2 | \Delta\Theta \right\rangle \ , \tag{6.3}$$

where $\langle |\nabla\Theta|^2 | \Delta\Theta \rangle$ is the average of $|\nabla\Theta|^2$ *conditional* on a given value of $\Delta\Theta$. $P(\Delta\Theta)$ is the probability to observe a given value of $\Delta\Theta$. Eq.(6.3) is exact. The question now is what is the dependence of $\langle |\nabla\Theta|^2 | \Delta\Theta \rangle$ on $\Delta\Theta$. In order to recover our result (5.5) this conditional average *must* satisfy

$$\left\langle |\nabla\Theta|^2 | \Delta\Theta \right\rangle = C(\Delta\Theta)^2 / S_2(R). \tag{6.4}$$

This means that the coefficient C_{2n} in (5.5) is n-independent. We can determine this coefficient from the particular case $n = 1$. Since $J_2 = \bar{\epsilon}$ the conclusion is that $C_n = 1$. Equipped with this we consider again the balance equation (1.8)

$$R^{1-d} \frac{\partial}{\partial R} R^{d-1} h(R) \frac{\partial}{\partial R} S_{2n}(R) = J_{2n}(R) \ . \tag{6.5}$$

Using now the definition of the scaling exponents $S_{2n}(R) \sim R^{\zeta_{2n}}$ we retrieve from (6.5) the result (1.12) which can be also written as

$$\zeta_{2n} = \frac{1}{2} \left[\zeta_2 - d + \sqrt{(\zeta_2 + d)^2 + 4\zeta_2(n-1)} \ \right] . \tag{6.6}$$

This is Kraichnan's anomalous scaling.

7. SUMMARY AND CONCLUSIONS

The first conclusion of this paper is that renormalized perturbation theory for hydrodynamic fields has the potential to describe nonperturbative effects. Exact resummations of the diagrammatic series result in exact equations for the

statistical quantities that contain not only perturbative but also nonperturbative effects.

Secondly, the passive scalar problem with rapidly decorrelating velocity field displays a particularly simple analytic structure in which all the scaling properties emanate from one differential operator $\hat{\mathcal{B}}(R)$. As a result this theory becomes "critical" in the sense that the anomalous exponent Δ that was introduced in paper I is exactly critical. The reason for this is that both Δ and Δ_c come from the same operator, and thus they must be the same. In such a situation the subcritical scenario that was suggested in [8] is untenable in this case.

Thirdly, the simplicity of the theory allowed us to compute the whole spectrum of anomalous exponents that are associated with the ultraviolet divergence. It was shown that these exponents are related to the spectrum β_l (3.11) which appears in the law of isotropization of the 2-point correlation function. The same exponents determine the R dependence of the correlation functions $L_{ll',m}$ that were introduced in (4.29).

Finally we showed that as far as the structure functions $S_n(R)$ are concerned, there are only two possibilities. Either the exponents satisfy $\zeta_{2n} = n\zeta_2$ and the scaling is normal or the scaling is anomalous with the outer renormalization scale and with the law (6.6). We noted that normal scaling also implies that the dissipation field is not mixing.

Whether or not the "critical" situation of this model with a marginal scaling exponent $\Delta = \Delta_c$ is structurally stable is an extremely important question that must await further research.

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APPENDIX A: DERIVATION OF THE EQUATION FOR THE SIMULTANEOUS TWO-POINT CORRELATOR

Here we obtain the equation of motion for the simultaneous two-point correlator, $\mathcal{F}(\mathbf{r}_1, \mathbf{r}_2, t = 0)$. We make use of Eq. (2.18) to verify that the operator

$$\mathcal{G}_{12}^{-1}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, t) \equiv \partial_t + \hat{\mathcal{D}}_1(\mathbf{r}_1 - \mathbf{r}_0) + \hat{\mathcal{D}}_1(\mathbf{r}_2 - \mathbf{r}_0) \tag{A1}$$

is the inverse of the product of Green's functions, that is,

$$\begin{aligned} &\mathcal{G}_{12}^{-1}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2, t)\mathcal{G}(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_1', t)\mathcal{G}(\mathbf{r}_0|\mathbf{r}_2, \mathbf{r}_2', t) \\ &= \delta(\mathbf{r}_1 - \mathbf{r}_1')\delta(\mathbf{r}_2 - \mathbf{r}_2')\delta(t). \end{aligned} \tag{A2}$$

We have used here the fact that the Green's function when multiplied by $\delta(t)$ may be evaluated at $t = 0$. Applying this operator and performing the spatial integrations, one obtains directly

$$\left[\hat{\mathcal{D}}_1(\mathbf{r}_1 - \mathbf{r}_0) + \hat{\mathcal{D}}(\mathbf{r}_2 - \mathbf{r}_0)\right]\mathcal{F}(\mathbf{r}_1, \mathbf{r}_2) = \Phi(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2) + \Phi_0(\mathbf{r}_1 - \mathbf{r}_2). \tag{A3}$$

Now using the definition of $\Phi(\mathbf{r}_0|\mathbf{r}_1, \mathbf{r}_2)$ from Eq.(2.12) and dropping the $\mathbf{r}_0$ dependence, one may rewrite this as

$$-\left[\kappa(\nabla_1^2 + \nabla_2^2) + h_{ij}(\mathbf{r}_1 - \mathbf{r}_2)\frac{\partial}{\partial r_{1i}}\frac{\partial}{\partial r_{2j}} + \hat{\mathcal{H}}(\mathbf{r}_1, \mathbf{r}_2)\right]\mathcal{F}(\mathbf{r}_1, \mathbf{r}_2) = \Phi_0(\mathbf{r}_1, \mathbf{r}_2), \tag{A4}$$

where the operator $\hat{\mathcal{H}}$ is given by

$$\hat{\mathcal{H}}(\mathbf{r}_1, \mathbf{r}_2) \equiv \left(h_{ij}(\mathbf{r}_1)\frac{\partial}{\partial r_{1i}} + h_{ij}(\mathbf{r}_2)\frac{\partial}{\partial r_{2i}}\right)\left(\frac{\partial}{\partial r_{1j}} + \frac{\partial}{\partial r_{2j}}\right). \tag{A5}$$

In the case that all quantities are only functions of $\mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2$, $\hat{\mathcal{H}}$ vanishes. The remaining terms are equivalent to the definition of the operator $\hat{\mathcal{D}}_2(\mathbf{R})$ in Eq.(2.32) and one recovers Eq.(2.29).

APPENDIX B: EQUATION FOR THE CUMULANT OF THE $2N$-POINT CORRELATOR

In this appendix we derive equations for the many-time correlators using the diagrammatic approach. The infinite diagrammatic series for the 4-point correlator was presented in [4]. Let us consider the series for the correlator. A typical diagram for a $2n$-point correlator consists of n 'tramways': strings of Green's functions, connected in pairs at one end by two-point correlators. These tramways may be interconnected by velocity correlators. Therefore to build up an arbitrarily complex diagram, one adds successively connections between chosen pairs of tramways. Let us consider a given pair of end-points, x_α and x_β. Group all diagrams together in which the Green's function beginning at x_α and that beginning at x_β are linked by a velocity correlator. The series of diagrams to the right of this first velocity correlator is again the series of diagrams for the full correlator.

The difficulty with writing a resummed equation for the full correlator is that the lowest order terms are Gaussian, and disconnected. In building upon the Gaussian terms by the addition of 'rungs' as just described, one will generate, amongst other terms, a series in which the disconnected parts remain disconnected, and may be resummed again into the Gaussian decomposition. This means that these terms actually appear twice. In order to avoid this one may write an an equation for the *cumulants*. For the cumulant the lowest order terms are those in which all tramways have one velocity correlator connection to another.

To simplify the appearance of this equation we introduce the operator $\widehat{C}_{\alpha\beta}$ which represents the addition of a rung, and here operates on $\mathcal{F}_{2n}(0|x_1, ..., x_\alpha, x_\beta, ..., x_{2n})$. The definition is

$$\widehat{C}_{\alpha\beta} * \mathcal{F}_{2n}(0|x_1, ..., x_\alpha, x_\beta, ..., x_{2n})$$

$$\equiv \int d\mathbf{r}_1' d\mathbf{r}_2' \int_{t_m}^{\infty} dt \mathcal{G}_2^0(0|x_\alpha, x_\beta, \mathbf{r}_1', t, \mathbf{r}_2', t) \tag{B1}$$
$$\times H_{ij}(\mathbf{r}_1', \mathbf{r}_2') \frac{\partial}{\partial r_1'} \frac{\partial}{\partial r_2'} \mathcal{F}_{2n}(0|x_1, ..., \mathbf{r}_1', t, \mathbf{r}_2', t, ..., x_{2n}).$$

Since $\mathcal{F}_{2n}(0|x_1, x_2, .., x_{2n})$ is symmetric with respect to all exchanges of coordinates this definition is sufficient for any pair of indices $1 \le \alpha, \beta \le 2n$.

The lowest order terms may be expressed in terms of the operator $\hat{C}$ and two-point correlators as

$$\mathcal{F}^c_{2n,0}(0|x_1, x_2, .., x_{2n}) = \prod_{j=2,4}^{2n} \hat{C}_{j,j+1} * \prod_{k=1,3..}^{2n-1} \mathcal{F}(x_k, x_{k+1}) \tag{B2}$$

Then the resummed equation has the form

$$\mathcal{F}^c_{2n}(0|x_1, x_2, .., x_{2n}) = \sum_{\text{perm}} \mathcal{F}^c_{2n,0}(0|x_1, x_2, .., x_{2n})$$
$$+ \sum_{\alpha>\beta} \hat{C}_{\alpha\beta} * \mathcal{F}^c_{2n}(0|x_1, x_2, .., x_{2n}). \tag{B3}$$

Now operate on both sides of the equation with the product of the inverse Green's functions (2.26). Using the fact that

$$\left(\frac{\partial}{\partial t_\alpha} + \hat{\mathcal{D}}_1(\mathbf{r}_\alpha)\right)\left(\frac{\partial}{\partial t_\beta} + \hat{\mathcal{D}}_1(\mathbf{r}_\beta)\right) C_{\alpha\beta} = \delta(t_\alpha - t_\beta)\mathcal{B}_{\alpha\beta}, \tag{B4}$$

one finds

$$\prod_{k=1}^{2n}\left(\frac{\partial}{\partial t_k} + \hat{\mathcal{D}}_1(\mathbf{r}_k)\right)\mathcal{F}^c_{2n}(0|x_1, x_2, .., x_{2n}) =$$
$$\sum_{\langle\alpha,\beta\rangle} \sum_{\substack{(\gamma_i) \\ \text{perm}(\gamma_1\neq\alpha,\beta)}} \left[\prod_{i=1}^{2n-2}\delta(t_{\gamma_i} - t_{\gamma_{i+1}})\right]\left(\frac{\partial}{\partial t_\alpha} + \hat{\mathcal{D}}_1(\mathbf{r}_\alpha)\right)\left(\frac{\partial}{\partial t_\beta} + \hat{\mathcal{D}}_1(\mathbf{r}_\beta)\right)\mathcal{B}_{\gamma_1,\gamma_2}\mathcal{F}(x_\alpha, x_{\gamma_1})$$
$$\left\{\prod_{j=2,4..}^{2n-4} \mathcal{B}_{\gamma_{j+1},\gamma_{j+2}}\mathcal{F}(\mathbf{r}_{\gamma_j}, \mathbf{r}_{\gamma_{j+1}})\right\}\mathcal{B}_{\gamma_{j+1},\gamma_{j+2}}\mathcal{F}(x_{\gamma_{2n-2}}, x_\beta)$$
$$+\sum_{\alpha>\beta}\delta(t_\alpha - t_\beta)\prod_{\gamma\neq\alpha,\beta}\left(\partial_t + \hat{\mathcal{D}}_1(\mathbf{r}_\gamma)\right)\mathcal{B}_{\alpha\beta}\mathcal{F}^c_{2n}(0|x_1, x_2, .., x_{2n}). \tag{B5}$$

Therefore we have derived a closed equation for the time development of the many-time cumulants of the $2n$-point correlator in terms of the operator $\mathcal{B}$ and the two-point moments only. We will not solve this equation in this paper.

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